Continued Fractions and Hyper Geometric Series with Bailey's Transformation and it's Applications

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Abstract: Euler developed the theory of continued fractions in the 1730's, driven in part by computational interests. It is perhaps a surprise that the association of a given series with a continued fraction first received full development in the case of the divergent series. This connection was born during Euler's correspondence with Nicholas Bernoulli from 1742 to 1745. While he had already used divergent series to find values of convergent numerical series, in his letters with Bernoulli his concepts and practices are exposed, if not developed. In this paper author try to establish transformation formulas using hyper geometric functions. In order to derive these transformations, two well-known methods are used i.e., the q-series and q-continued factions. Main objective is to establish transformation formulas using hyper geometric functions with the help of known transformations formulas in hyper geometric functions. The Bailey's transform is obtained from a suitably modified G terminating very well poised summation theorem and term wise transformation. It is been interpreted as a matrix inversion result of two infinite, lower triangular matrixes. This provides a higher dimensional generalization of Andrew's matrix inversion formulation of Bailey's transformation.

Keywords: Continued fraction, Bailey's Transformation, q-series and q-function.

I. Continued Fraction

From the early age of mathematics Continued fractions have been playing a very important role in number theory and classical analysis from the time of Euler and Gauss. Generalized hyper geometric series, both q-series and q-fractions, have been a very significant tool in the derivation of continued fraction representations.

II. Bailey transformation

In 1951 and 1952, L. J. Slater derived one hundred and thirty identities of Rogers-Ramanujan type with the help of Bailey transformation formula.

1. If we take $\alpha_r = Z^r$ and $\delta_r = Z^r$

then

$$\beta_n = \sum_{r=0}^{n} z^r = \frac{1-z^{n+1}}{1-z}, \quad |z| < 1$$

$$\gamma_n = \sum_{r=0}^{\infty} z^r = \frac{1}{1-z}$$

Putting the value of α_n, δ_n, β_n and Υ_n in equation, then we get

$$\sum_{n=0}^{\infty} z^n \left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} z^n \left(\frac{1-z^{n+1}}{1-z}\right)$$

$$\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} (z^n - z^{2n+1})$$

1. If we take

$$\alpha_r = \frac{(a)_r(1+a/2)_r(b)_r\,(c)_r(-m)_r}{(a/2)_r(1+a-b)_r(1+a-c)_r(1+a+m)_r r!}$$

Then

$$\beta_n = \sum_{r=0}^{n} \frac{(a)_r(1+a/2)_r(b)_r\,(c)_r(-m)_r}{(a/2)_r(1+a-b)_r(1+a-c)_r(1+a+m)_r r!}$$

$$\beta_n = \frac{(1+a)_m(1+a-b-c)_m}{(1+a-b)_m(1+a-c)_m}$$

And

$$\delta_r = Z^r$$

Then

$$\gamma_n = \sum_{r=0}^{\infty} z^r = \frac{1}{1-z}$$

Putting the value of α_n, δ_n, β_n and Υ_n in equation, then we get

$$\sum_{n=0}^{\infty} \frac{(a)_n(1+a/2)_n (b)_n (c)_n(-m)_n}{(a/2)_n(1+a-b)_n(1+a-c)_n(1+a+m)_n n!} = (1-z)\sum_{n=0}^{\infty} \frac{(1+a)_n(1+a-b-c)_n}{(1+a-b)_n(1+a-c)_n} z^n$$

2. **If we take**

$$\alpha_r = \frac{(a)_r(1+a/2)_r(b)_r\,(c)_r(-1)^r}{(a/2)_r(1+a-b)_r(1+a-c)_r\,r!}$$

Then

$$\beta_n = \sum_{r=0}^{n} \frac{(a)_r(1+a/2)_r(b)_r\,(c)_r(-1)^r}{(a/2)_r(1+a-b)_r(1+a-c)_r r!}$$

$$\beta_n = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}$$

And

$$\delta_r = Z^r$$

Then

$$\gamma_n = \sum_{r=0}^{\infty} z^r = \frac{1}{1-z}$$

Putting the value of α_n, δ_n, β_n and Υ_n in equation, then we get

$$\sum_{n=0}^{\infty} \frac{(a)_n(1+a/2)_n (b)_n (c)_n(-1)^n}{(a/2)_n(1+a-b)_n(1+a-c)_n n!} = (1-z)\frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} z^n$$

Provided that Re(b+c-a/2)<1.

3. **If we take**

$$\alpha_r = \frac{(a)_r(1+a/2)_r(b)_r\,(-m)_r(-1)^r}{(a/2)_r(1+a-b)_r(1+a+m)_r\,r!}$$

Then

$$\beta_n = \frac{(1+a)_m}{(1+a-b)_m}$$

And $\delta_r = Z^r$

Then

$$\gamma_n = \sum_{r=0}^{\infty} z^r = \frac{1}{1-z}$$

Putting the value of α_n, δ_n, β_n and Υ_n in equation, then we get

$$\sum_{n=0}^{\infty} \frac{(a)_n (1+a/2)_n\,(b)_n\,(-m)_n (-1)^n}{(a/2)_n (1+a-b)_n (1+a+m)_n n!} = (1-z) \sum_{n=0}^{\infty} \frac{(1+a)_n}{(1+a-b)_n} z^n$$

4. If we take

$$\alpha_r = \frac{(a)_r\,(b)_r\,(c)_r}{(1+a-b)_r\,(1+a-c)_r\,r!}$$

Then

$$\beta_n = \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{a}{2}-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(1+\frac{a}{2}-c\right)\Gamma(1+a-b-c)}$$

And $\delta_r = Z^r$

Then

$$\gamma_n = \sum_{r=0}^{\infty} z^r = \frac{1}{1-z}$$

Putting the value of α_n, δ_n, β_n and Υ_n in equation, then we get

$$\sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,(c)_n}{(1+a-b)_n\,(1+a-c)_n\,n!} = \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{a}{2}-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(1+\frac{a}{2}-c\right)\Gamma(1+a-b-c)}(1-z) \sum_{n=0}^{\infty} z^n$$

Let $(a;q)_n$ be the q-Pochhammer symbol, let a be an indeterminate, and let the lower triangular matrices $F=(f)_{n\,k}$ and $G=(g)_{n\,k}$ be defined as

$$f_{n\,k}=\frac{1}{(q;q)_{n-k}\,(a\,q;q)_{n+k}}$$

and

$$g_{n\,k}=\frac{\left(1-a\,q^{2\,n}\right)(a;q)_{n+k}}{(1-a)\,(q;q)_{n-k}}\,(-1)^{n-k}\,q^{\binom{n-k}{2}}.$$

Then F and G are matrix inverses.

III. Applications

The theory of continued fractions has applications in cryptographic problems and in solution methods for Diophantine equations. We will first examine the basic properties of continued fractions such as convergents and approximations to real numbers. Then we will discuss a computationally efficient attack on the RSA cryptosystem (Wiener's attack) based on continued fractions.

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