International Journal of Advanced and Innovative Research (2278-7844)/

Volume 8 Issue 6

Certain Continued Fractions involving Generalized Lambert Series

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***Abstract*— The object of this paper is to establish some interesting results on continued fraction for the ratio of two generalized Lambert series using of known identities due to Ramanujan.**
***Keywords*— Generalized Lambert Series, Basic hyper geometric series, Continued fractions.**

I. Introduction

For real or complex $a, q < 1$, the q-shifted factorial is defined as

$$(a,q)_n = \begin{cases} 1 & if\ n=0; \\ (1-a)(1-aq)(1-aq^2),...,(1-aq^{n-1}) & if\ n \in N. \end{cases} \quad (1)$$

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r) \quad (2)$$

and

$$(a_1,a_2,...,a_r;q)_n = \prod_{k=1}^{r}(a_k;q)_n \quad (3)$$

Lambert series is a very well known series in analytic function theory and Number theory. An infinite series of the form

$$\sum_{n=1}^{\infty} a_n \frac{q^n}{1-q^n} \quad (4)$$

is called the Lambert series. This series is connected with the convergence of the power series.

If $\sum_{n=1}^{\infty} a_n$ converges, then the Lambert series (4) converges for all values of q except for q = ±1, otherwise it converges for those values of q for which the series $\sum_{n=1}^{\infty} a_n q^n$ converges.

We shall use the following identities due to Ramanujan mentioned in Andrews and Berndt [2] on pages 252 and 119.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \frac{G(q^4)}{(q;q^2)_\infty} \quad (5)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n+1}} = \frac{H(-q)}{(q;q^2)_\infty} \quad (6)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}} = \frac{G(-q)}{(q;q^2)_\infty} \quad (7)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q;q)_{2n+1}} = \frac{H(q^4)}{(q;q^2)_\infty} \quad (8)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2}}{(q^4;q^4)_n(-q;q^2)_n} = \frac{G(q)}{(-q;q)_\infty} \quad (9)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2-2n}}{(q^4;q^4)_n(-q;q^2)_n} = \frac{H(q)}{(-q;q)_\infty} \quad (10)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+2n}}{(q^4;q^4)_n(-q;q^2)_n} = \frac{H(q)}{(-q;q)_\infty} \quad (11)$$

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+2}} = (q^5;q^5)_\infty^2 G(q) \quad (12)$$

$$\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}} = (q^5;q^5)_\infty^2 H(q) \quad (13)$$

$$\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{5n+1}}=(q^5;q^5)_\infty^2\frac{G^2(q)}{H(q)} \quad (14)$$

$$\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{5n+2}}=(q^5;q^5)_\infty^2\frac{H^2(q)}{G(q)} \quad (15)$$

$$\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{5n+1}}=(q^5;q^5)_\infty^2 G(q) \quad (16)$$

$$\sum_{n=-\infty}^{\infty}\frac{q^{n}}{1-q^{5n+3}}=(q^5;q^5)_\infty^2 H(q) \quad (17)$$

$$\sum_{n=-\infty}^{\infty}q^{5n^2+2n}\frac{1+q^{5n+1}}{1-q^{5n+1}}=(q^5;q^5)_\infty^2\frac{G^2(q)}{H(q)} \quad (18)$$

$$\sum_{n=-\infty}^{\infty}q^{5n^2+4n}\frac{1+q^{5n+2}}{1-q^{5n+2}}=(q^5;q^5)_\infty^2\frac{H^2(q)}{G(q)} \quad (19)$$

$$\sum_{n=-\infty}^{\infty}\frac{q^{4n}}{1-q^{10n+1}}=(q^5;q^5)_\infty^2 G(q) \quad (20)$$

$$\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{10n+3}}=(q^5;q^5)_\infty^2 H(q) \quad (21)$$

where

$$H(q)=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \quad (22)$$

and

$$G(q)=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty} \quad (23)$$

$$\frac{H(q)}{G(q)}=\frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty}=\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\ldots} \quad (24)$$

Roger's Fine Identity is given as

$$\sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(\beta;q)_n}\tau^n=\sum_{n=0}^{\infty}\frac{(\alpha;q)_n(\frac{\alpha\tau q}{\beta};q)_n\beta^n\tau^n q^{n^2-n}\left(1-\alpha\tau q^{2n}\right)}{(\beta;q)_n(\tau;q)_{n+1}} \quad (25)$$

In the above equation (25) replacing β by αq and then replacing q by q^k and $\tau = q^i$, $\alpha = q^j$ we will have the following equation:

$$\sum_{n=0}^{\infty}\frac{q^{in}}{\left(1-q^{kn+j}\right)}=\sum_{n=0}^{\infty}\frac{q^{kn^2+(i+j)n}\left(1-q^{2kn+i+j}\right)}{\left(1-q^{kn+i}\right)\left(1-q^{kn+j}\right)} \quad (26)$$

For j = i, equation (26) will give the following equation:

$$\sum_{n=0}^{\infty}\frac{q^{in}}{\left(1-q^{kn+i}\right)}=\sum_{n=0}^{\infty}\frac{q^{kn^2+2in}\left(1+q^{kn+i}\right)}{\left(1-q^{kn+i}\right)} \quad (27)$$

II. MAIN RESULT

Dividing (10) by (9) and using (24) we have the following continued fraction:

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2-2n}}{(q^4;q^4)_n(-q;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2}}{(q^4;q^4)_n(-q;q^2)_n}}=\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\ldots} \quad (28)$$

Dividing (11) by (9) and using (24) we have the following continued fraction:

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2+2n}}{(q^4;q^4)_n(-q;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2}}{(q^4;q^4)_n(-q;q^2)_n}}=\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\ldots} \quad (29)$$

Dividing (8) by (5) and using (24) we have the following continued fraction:

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q;q^2)_{2n}\left(1-q^{2n+1}\right)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_{2n}}}=\frac{1}{1+}\frac{q^4}{1+}\frac{q^8}{1+\ldots} \quad (30)$$

Dividing (4) by (3) and using (24) we have the following continued fraction:

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q^2)_{2n}}}{\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q^2)_{2n}\left(1-q^{2n+1}\right)}}=\frac{1}{1-}\frac{q}{1+}\frac{q^2}{1-}\frac{q^3}{1+\ldots} \quad (31)$$

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Dividing (10) by (9) and using (26) we have the following continued fraction:

$$\frac{\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+2})(1-q^{5n+4})}}{\sum_{n=0}^{\infty}\frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+1})(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+4})(1-q^{5n+3})}}$$

$$=\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\ldots} \qquad (32)$$

Dividing (10) by (11) and using (26) we have the following continued fraction:

$$\frac{\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+2})(1-q^{5n+4})}}{\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1-q^{10n+2})}{(1-q^{5n+1})^2}-\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+4})}{(1-q^{5n+4})}} \qquad (33)$$

$$=\left[\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\ldots}\right]^2$$

Dividing (12) by (9) and using (26) we have the following continued fraction:

$$\frac{\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+3})}{(1-q^{5n+3})}}{\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1-q^{10n+2})}{(1-q^{5n+1})^2}-\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+4})}{(1-q^{5n+4})}} \qquad (34)$$

$$=\left[\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\ldots}\right]^3$$

Dividing (21) by (20) and using (26) we have the following continued fraction:

$$\frac{\sum_{n=0}^{\infty}\frac{q^{10n^2+5n}(1-q^{20n+5})}{(1-q^{10n+3})(1-q^{10n+2})}-\sum_{n=0}^{\infty}\frac{q^{10n^2+15n+5}(1+q^{20n+15})}{(1-q^{10n+7})(1-q^{10n+8})}}{\sum_{n=0}^{\infty}\frac{q^{10n^2+5n}(1-q^{20n+2})}{(1-q^{10n+1})(1-q^{10n+4})}-\sum_{n=0}^{\infty}\frac{q^{10n^2+15n+5}(1+q^{20n+15})}{(1-q^{10n+9})(1-q^{10n+6})}} \qquad (35)$$

$$=\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\ldots}$$

Dividing (15) by (18) and using (26) we have the following continued fraction:

$$\frac{\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})}}{\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+3})}{(1-q^{5n+3})}} \qquad (36)$$

$$=\left[\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\ldots}\right]^{-3}$$

Dividing (12) by (13) and using (26) we have the following continued fraction:

$$\frac{\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+2})(1-q^{5n+4})}}{\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1-q^{10n+2})}{(1-q^{5n+1})^2}-\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1-q^{5n+4})}{(1-q^{5n+4})}} \qquad (37)$$

$$=\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\ldots}$$

ACKNOWLEDGMENT

The authors are thankful to Dr.S.N.Singh, Ex. reader and Head, Department of Mathematics, T.D.P.G. College, Jaunpur (U.P.), INDIA,for his noble guidance during the preparation of this paper.

CONCLUSION

The authors have established continued fractions for the ratio of two generalized Lambert series (results 28 to 31).Results 32 to 37 are continued fractions for the ratio of difference of two generalized Lambert series. These results may be useful to the researchers for their future references. Some special cases of these results may be derived.

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